R_s	$\left(\frac{d^2f_1}{d\eta^2}\right)_{\eta=0}$	$\left(\frac{d^4f_1}{d\eta^4}\right)_{\eta=0}$	$\left(\frac{d^2f_2}{d\eta^2}\right)_{\eta=0}$	$\left(\frac{d^4f_2}{d\eta^4}\right)_{\eta=0}$	$\left(\frac{d^2f_1}{d\eta^2}\right)_{\eta=1}$	$\left(\frac{d^2f_2}{d\eta^2}\right)_{\eta=1}$
−1·0	4·24	−30·5	−0·945	26·4	−7·41	−3·01
−5·0	4·90	−61·6	−2·97	152·0	−6·30	−3·91
−10·0	5·54	−109·0	−6·98	661·0	−6·23	−4·16

The dimensionless velocity gradient at the wall is given by

$$\left(\frac{\partial q_T}{\partial \eta}\right)_{\eta=1} = \frac{1}{4}\left(\frac{d^2f_1}{d\eta^2}\right)_{\eta=1} + \frac{\epsilon^2}{4\xi^2}\left(\frac{d^2f_2}{d\eta^2} - \frac{1}{2}\frac{d^2f_1}{d\eta^2}\right)_{\eta=1} + \ldots$$

The wall shear stress, which is proportional to the negative value of this quantity, is seen to be independent of ξ to the first approximation and decreases slightly from $R_s = -1{\cdot}0$ to $-10{\cdot}0$ (see the table). The effect of the second-order modification is generally small and diminishes with the increase of distance ξ. Thus, as in the hypersonic small-disturbance theory, the first-order approximation is expected to describe rather accurately the flow conditions in a porous paraboloidal pipe.

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THE HEAT BALANCE INTEGRAL METHOD*

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INTRODUCTION

APPROXIMATE solutions to transient diffusion problems may be obtained relatively easily by the use of what is commonly called "The Heat Balance Integral Method." THEBIM. THEBIM is applicable to one-dimensional linear and non-linear problems involving temperature dependent thermal properties [6, 7, 18], non-linear boundary conditions [7, 9], and phase change problems such as freezing [4, 5, 7–11, 17]. The applicability to phase change problems is of special importance [1, 2, 16] because existing closed form solutions to these significant problems are highly restrictive as to allowable initial conditions and boundary conditions [3, 12–15].

The accuracy of an approximate solution is in general unknown [2, 5–8, 11, 16]. Using THEBIM, attempts to increase the accuracy of an approximate solution have sometimes actually caused a decrease in accuracy [6, 7, 16]. There may therefore be some value in an accuracy criterion which can be easily used even when the exact solution is unknown. The use of such a criterion is illustrated here for a classical problem.

A SAMPLE PROBLEM

Let $T(x, t)$ be the temperature at position x at time t in a semi-infinite slab having constant thermal conductivity k,

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density ρ, specific heat c, and thermal diffusivity $a \equiv k/\rho c$. Also, let subscripts in x and t denote partial differentiation, so $T_{xt}(x, t) = \partial^2 T/\partial x \partial t$. Finally, let $q(x, t)$ be defined as the heat flow in the $+x$ direction per unit time per unit of transverse area. Then, from Fourier

$$q(x, t) = -kT_x(x, t), \qquad 0 \leqslant x, \qquad 0 \leqslant t. \tag{1}$$

For no internal heat sources, (1) and conservation of energy yield the usual partial differential equation

$$kT_{xx}(x, t) = \rho c T_t(x, t). \tag{2}$$

Initial and boundary conditions to be used are

$$T(x, 0) = T_{\text{initial}} = T_i = \text{constant}, \quad x > 0; \tag{3}$$

$$T(0, t) = T_{\text{surface}} = T_s = \text{constant}, \quad t > 0; \tag{4}$$

$$T(\infty, t) = T_{\text{initial}} = T_i = \text{constant}, \quad t \text{ finite}. \tag{5}$$

Equations (2)–(5) constitute a classical problem with solution [3, 16]

$$(T - T_i)/(T_s - T_i) = [1 - \text{erf}(\sqrt{(x^2/4at)})], \text{ where} \tag{6}$$

$$Q_e(t) \equiv q_{\text{exact}}(0, t) = -kT_x(0, t) = k(T_s - T_i)/\sqrt{(\pi at)}. \tag{7}$$

THE APPROXIMATION PROCEDURE

Assume there exists a function U with

$$U(x, t) \simeq T(x, t). \tag{8}$$

The *actual* temperature distribution $T(x, t)$ satisfies the partial differential equation (2) and also the following integral equation:

$$\int_0^t q(0, t')\,dt' = \int_0^t -kT_x(0, t')\,dt' = \int_0^\infty \rho c[T(x, t) - T(0, t)]\,dx. \tag{9a}$$

This is simply a "balance" of the heat energy input on the left against its measurable effect on the right; it may be called the "heat balance integral". Many approximation techniques (e.g. that of "lumped parameters") do not satisfy (2) but do satisfy (9a) [1, 16]. Here also it is not required that U satisfy (2) but it is required that U satisfy (9a) so

$$\int_0^t -kU_x(0, t')\,dt' = \int_0^\infty \rho c[U(x, t) - T_i]\,dx. \tag{9b}$$

Assume that, at any finite time t, the significant (measurable) effects of the boundary disturbance (3) and (4) do not penetrate beyond some finite distance $x = p(t)$. This assumption can be stated in the following mathematical form:

$$U(p(t), t) = T_i \quad \text{for} \quad t > 0; \tag{10a}$$

$$U(x, t) = T_i \quad \text{for} \quad x > p(t). \tag{10b}$$

Use of (10b) in (9b) yields the following form of the heat balance integral:

$$\int_0^t -kU_x(0, t')\,dt' = \int_0^{p(t)} \rho c[U(x, t) - T_i]\,dx. \tag{11}$$

Equation (11) will be used in place of (2). In place of the initial condition (3), the appropriate initial condition is

$$p(0) = 0. \tag{12}$$

Also, the boundary conditions (4, 5) on $T(x, t)$ are now replaced by the following boundary conditions on $U(x, t)$.

$$U(0, t) = T_s \quad \text{for} \quad t > 0; \tag{13}$$

$$U(p(t), t) = T_i \quad \text{for} \quad t > 0. \tag{14}$$

The approximation procedure is now reduced to finding a function $U(x, t) \simeq T(x, t)$ such that (13) and (14) are satisfied, this function U then being used in (11) to find the penetration depth $p(t)$, subject to the initial condition (12). In finding $U(x, t)$, it is convenient (but not necessary [7]) to restrict the search to polynomial functions of the form

$$U_n(x, t) = A_n(t) + B_n(t).x + C_n(t).x^2 + \ldots x^n, \tag{15}$$

where the coefficients are functions of time, and where the corresponding value of $p(t)$ will be denoted by $p_n(t)$.

RESULTS

The first possible approximation of the form (15) which can satisfy the *two* conditions (13) and (14) is

$$U_1(x, t) = A_1(t) + B_1(t).x. \tag{16}$$

Use of (16) with (13, 14) yields

$$[(U_1 - T_i)/(T_s - T_i)] = [1 - x/p_1(t)]. \tag{17}$$

To find $p_1(t)$, it is necessary to use (17) in (11), yielding

$$-k(T_i - T_s)\int_0^t [dt'/p_1(t')] = \rho c(T_s - T_i)p_1(t)/2. \tag{18}$$

Differentiation and rearrangement yields

$$p_1.(dp_1/dt) = 2a = \text{constant}. \tag{19}$$

Integration and use of the initial condition (12) yields

$$p_1(t) = \sqrt{(4at)}. \tag{20}$$

The first approximation $U_1(x, t)$ is fully defined by (17) and (20). Note that

$$Q_1(t) \equiv -kU_{1_x}(0, t) = k(T_s - T_i)/p_1 = k(T_s - T_i)/\sqrt{(4at)}; \tag{21}$$

$$M_1 \equiv [Q_1(t)/Q_e(t) - 1] = -0{\cdot}114 = -11{\cdot}4 \text{ per cent}. \tag{22}$$

HIGHER APPROXIMATIONS

It is possible to increase the degree of the polynomial approximation $U_n(x, t)$. The degree n that can be used

increases with the number of conditions such as (13) and (14) available to determine the coefficients of $U_n(x, t)$. Extra conditions cannot be obtained by simple differentiation of the boundary conditions (13) and (14) because such differentiation yields two conditions already satisfied identically by $U_1(x, t)$, namely

$$U_t(0, t) = 0, \tag{23}$$

and

$$U_x(p(t), t).(\mathrm{d}p/\mathrm{d}t) + U_t(p(t), t) = 0. \tag{24}$$

It is however possible to require that $U(x, t)$ satisfy the heat conduction equation (2) at $x = 0$ or $x = p(t)$ or both. Use of (2) in (23) and (24) yields

$$U_{xx}(0, t) = 0, \tag{25}$$

and

$$U_x(p(t), t).(\mathrm{d}p/\mathrm{d}t) + aU_{xx}(p(t), t) = 0. \tag{26}$$

Let $U_{2L}(x, t)$ and $U_{2R}(x, t)$ be the second degree polynomial approximations corresponding respectively to the use of the extra conditions (25) and (26) at the left end ($x = 0$) and right end ($x = p$) of the domain, with each approximation also satisfying (11)–(15). Results are in Table 1, and it may be noted that U_{2R} does not seem to have been published previously.

An alternate second degree approximation can be obtained by requiring that the approximation $U(x, t)$ satisfy the heat conduction equation (2) on an *algebraic average* basis over the domain $0 \leqslant x \leqslant p(t)$. Mathematically, this domain condition can be stated thus:

$$\int_0^{p(t)} [kU_{xx}(x, t) - \rho cU_t(x, t)]\,\mathrm{d}x = 0, \tag{27}$$

whence

$$kU_x(p(t), t) - kU_x(0, t) = \int_0^{p(t)} \rho cU_t(x, t)\,\mathrm{d}x. \tag{28}$$

But, from (14), $U(p(t), t) = T_i$ = constant so (28) can be written

$$kU_x(p(t), t) - kU_x(0, t) = \frac{\mathrm{d}}{\mathrm{d}t}\left\{\int_0^{p(t)} \rho c[U(x, t) - T_i]\,\mathrm{d}x\right\}. \tag{29}$$

Differentiation of (11) with respect to time and use in (29) then yields the following as the *simplified form of the new domain condition* (27):

$$kU_x(p(t), t) = 0. \tag{30}$$

Let $U_2(x, t)$ be the second degree polynomial approximation corresponding to the use of (27) in lieu of (25) or (26). Results are in Table 1.

"Higher" approximations can be obtained by requiring that $U(x, t)$ satisfy two or three of the conditions (25)–(27). Results are in Table 1.

*Table 1. Approximate temperature distributions U_n in order of decreasing e_n**

n	M_n (%)	e_n	b_n	$u_n(y) \equiv \dfrac{U_n(x, t) - T_i}{T_s - T_i}$, $y \equiv \dfrac{x}{p}$	Conditions used to obtain $U_n(x, t)$
1	−11·38	0·1667	4	$u_1 = (1 - y)$	11–15
2L	−11·38	0·1667	4	$u_{2L} = 1 - y + 0y^2$	11–15, 25
3LR	−39·58	0·0907	16	$u_{3LR} = 1 - \frac{15}{11}y + 0y^2 + \frac{4}{11}y^3$	11–15, 25, 26
3L	−6·00	0·0417	8	$u_{3L} = 1 - \frac{3}{2}y + 0y^2 + \frac{1}{2}y^3$	11–15, 25, 27
3R	+8·54	0·0233	24	$u_{3R} = (1 - y)^3 = 1 - 3y + 3y^2 - y^3$	11–15, 26, 27
2	+2·33	0·0192	12	$u_2 = (1 - y)^2 = 1 - 2y + y^2$	11–15, 27
2R	+1·13	0·0175	$\sqrt{(33)} + 3$	$u_{2R} = 1 + \left[\frac{\sqrt{(33)} - 1}{4}\right]y + \left[\frac{\sqrt{(33)} - 3}{4}\right]y^2$	11–15, 26
4	−2·92	0·0053		$u_4 = 1 - 2y + 0y^2 + 2y^3 - y^4$	11–15, 25, 26, 27

* Note: $M \equiv [U_x(0, t)/T_x(0, t) - 1]$ = dimensionless error in surface heat flux; $p = \sqrt{(b_n at)}$.

ACCURACY

The following is a calculable measure of the error inherent in the use of any approximation $U(x, t)$:

$$E(t) \equiv \int_0^{p(t)} [kU_{xx}(x, t) - \rho cU_t(x, t)]^2\,\mathrm{d}x \geqslant 0. \tag{31}$$

If $U(x, t)$ were the exact solution $T(x, t)$, then $E(t)$ would be identically zero by (2). The effect of the square under the integral is to prevent algebraic canceling of errors of opposite sign and to magnify the importance of space regions wherein $U(x, t)$ does not closely satisfy the heat conduction equation

(2). Let $E_n(t)$ be the result corresponding to $U_n(x, t)$, $p_n(t)$. Then

$$E_n(t) = e_n \cdot (k[T_s - T_i])^2 \cdot (at)^{-\frac{1}{2}}, \quad e_n = \text{constant}. \tag{32}$$

The lower the value of $E_n(t)$ or of e_n, the more accurate the solution is expected to be. This expectation appears to be borne out by a comparison of the $U_n(x, t)$ with $T(x, t)$.

A graph could be used to show the $U_n(x, t)$ and $T(x, t)$ as a function of the single independent variable $Y = (x/\sqrt{(4at)})$. The various curves on the graph would however lie very close to each other and would cross, making the graph difficult to interpret. In lieu of the graph, Table 2 presents the scaled error $[(U_n - T)/(T_s - T_i)]$ as a function of Y.

The relatively high accuracy of U_{2R} was not anticipated, but was easily recognized by virtue of the low value of e_{2R}. [Those involved in boundary layer analysis may wish to keep in mind that U_{2R} is a highly accurate solution which does *not* satisfy the domain condition (27) or, equivalently, the smoothing condition (30).]

In conclusion, use of the numerical error criterion $E(t)$ appears to make possible the straightforward evaluation of the relative accuracy of successive approximations even when the exact solution is not known.

Table 2. Error versus position when using an approximation U_n

$y = \frac{x}{\sqrt{(4at)}}$	$\frac{T - T_i}{T_s - T_i}$	Scaled error $= (U_n - T)/(T_s - T_i)$							$y = \frac{x}{\sqrt{(4at)}}$
		$n = 1, 2L$	$n = 3LR$	$n = 3L$	$n = 3R$	$n = 2$	$n = 2R$	$n = 4$	
0·0	1·0000	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000	0·0
0·1	0·8875	0·0124	0·0443	0·0065	−0·0050	0·0003	0·0014	0·0032	0·1
0·2	0·7772	0·0227	0·0867	0·0119	−0·0027	0·0050	0·0070	0·0060	0·2
0·3	0·6713	0·0286	0·1253	0·0152	0·0043	0·0122	0·0145	0·0081	0·3
0·4	0·5716	0·0283	0·1585	0·0154	0·0141	0·0198	0·0222	0·0089	0·4
0·5	0·4795	0·0205	0·1852	0·0122	0·0246	0·0264	0·0285	0·0082	0·5
0·6	0·3961	0·0038	0·2045	0·0056	0·0343	0·0310	0·0323	0·0059	0·6
0·7	0·3221	−0·0221	0·2161	−0·0040	0·0421	0·0328	0·0330	0·0021	0·7
0·8	0·2578	−0·0578	0·2199	−0·0159	0·0474	0·0316	0·0303	−0·0028	0·8
0·9	0·2030	−0·1030	0·2164	−0·0288	0·0500	0·0276	0·0245	−0·0084	0·9
1·0	0·1572	−0·1572	0·2063	−0·0411	0·0499	0·0213	0·0159	−0·0141	1·0
1·1	0·1197	*	0·1907	−0·0512	0·0474	0·0133	0·0053	−0·0191	1·1
1·2	0·0896	*	0·1706	−0·0570	0·0430	0·0046	−0·0063	−0·0229	1·2
1·3	0·0659	*	0·1475	−0·0564	0·0373	−0·0037	−0·0181	−0·0251	1·3
1·4	0·0477	*	0·1224	−0·0475	0·0309	−0·0109	−0·0291	−0·0253	1·4
1·5	0·0338	*	0·0967	*	0·0243	−0·0159	*	−0·0235	1·5
1·6	0·0236	*	0·0716	*	0·0180	−0·0178	*	−0·0201	1·6
1·7	0·0162	*	0·0480	*	0·0124	−0·0158	*	−0·0155	1·7
1·8	0·0109	*	0·0269	*	0·0077	*	*	−0·0109	1·8
1·9	0·0072	*	0·0091	*	0·0040	*	*	*	1·9
2·0	0·0046	*	−0·0046	*	0·0015	*	*	*	2·0
	e_n =	0·1667	0·0907	0·0417	0·0233	0·0192	0·0175	0·0053	e_n

* Note: Blank entries in the table correspond to $x > p_n(t)$. For blank entries, $U_n = T_i$ and the scaled error is exactly equal to the negative of the value shown in the second column above.

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